Hypocycloid Juggling Patterns

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Abstract

When points are distributed evenly on a hypocycloid path and animated along that path, they can be seen as clustering together into "wheels", groups of points that lie at the vertices of regular polygons and that rotate in synchrony. In some cases the points can group into two different sets of wheels, rotating in opposite directions. I derive formulas that predict the number of such wheels and the number of points on each one. The resulting animations are visually compelling and reminiscent of the motion of balls or clubs in multi-person juggling patterns.

1 Introduction

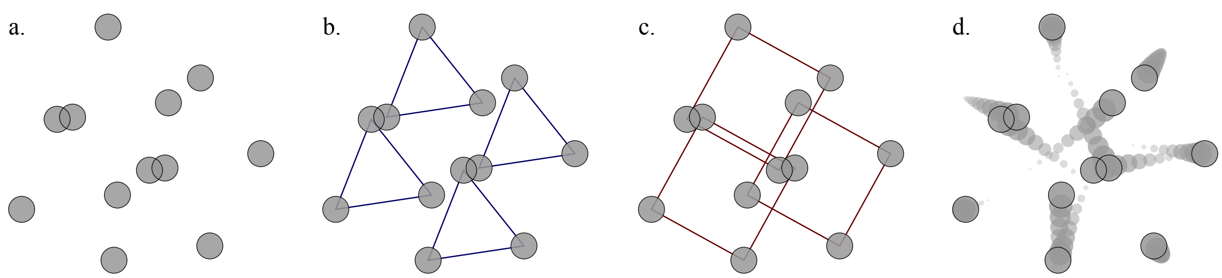

Figure 1: *A demonstration of a pattern of evenly spaced points on a hypocycloid, later to be referred to as a* $(3,7,12)$ *juggling pattern. The points are shown on their own in (a), then joined to show groupings of threes and fours in (b) and (c). In (d) the points are given "comet tails" to suggest motion. Please see the supplementary files for animated versions of the figures in this paper.*

At first glance, the arrangement of points in Figure 1(a) might not seem to have any structure. However, after staring at the points for a while one might be able to organize them into four groups, where the three points in each group lie at the vertices of congruent equilateral triangles, as shown in Figure 1(b). By a similar process, the points can also be collected into three groups of four as in Figure 1(c), where the groups form congruent squares.

The structure revealed above becomes most apparent—and most exciting—when the points are animated, as suggested by Figure 1(d). Obviously I cannot depict the motion faithfully in this document; please see the paper's supplementary files for animations, or visit an interactive version online at `http://isohedral.ca/hypocycloid-juggling-patterns/`. The points can be animated in such a way that they all travel counterclockwise around the centre of the picture, while simultaneously preserving the groupings into fixed-size, congruent equilateral triangles and squares. Moreover, as the points move they weave intricately around each other in a hypnotic dance. The effect is reminiscent of the motion of balls or clubs in a complex multi-person juggling pattern.

Although we might imagine constructing these patterns in an ad hoc manner by building a symmetric ring of symmetric rings of points, it turns out that the family of mathematical curves known as *hypocycloids* can serve

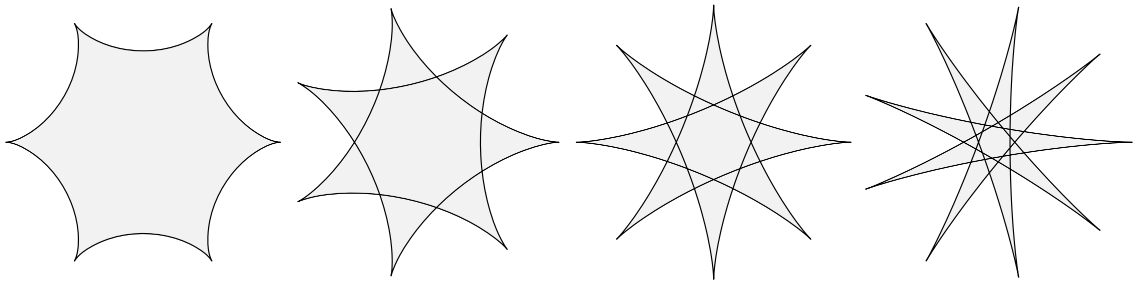

Figure 2 : *Drawings of a few simple rational hypocycloid curves based on gears of radius p/q rolling around the inside of a unit circle.*

as an elegant and rigorous mathematical framework for analyzing them, and for constructing new patterns with desired structures. In this paper I describe the use of hypocycloids in this context and give formulas that predict the formation of structures like those in Figure 1(b) and (c). Because of the use of hypocycloids, and the metaphorical visual connection to juggling, I refer to these animated patterns as "hypocycloid juggling patterns".

2 Hypocycloids

Let r and R be positive real numbers, with $r < R$. Imagine a small circle of radius r, which I call the "gear", rolling around the inner circumference of a large circle of radius R. By tracing the path of a point on the edge of the gear as it rolls, we obtain a curve known as a *hypocycloid* [4]. Up to similarity, the shapes of hypocycloids are determined entirely from a single parameter, the ratio r/R; henceforth we assume that $R = 1$.

When r is irrational (and assuming $R = 1$), the gear will be rotated by incommensurate amounts each time it returns to the same position in the large circle. In that case, the point will eventually trace out (a dense subset of) an annulus. The geometry and aesthetics become more interesting when r is rational; let us assume it is of the form p/q for integers $p < q$ in lowest terms. In that case, the hypocycloid will resemble a $\{q/p\}$ star polygon with curved edges: a symmetric shape with q cusps meeting the large circle. A few simple examples of rational hypocycloids are shown in Figure 2.

These rational hypocycloids are familiar objects in mathematical art. Most obviously, they belong to the more general family of *hypotrochoids*, in which the point being traced can be located at any distance from the centre of the small circle (in a hypocycloid, that distance is fixed at r). Hypotrochoids are precisely the curves traced by the classic Spirograph toy, where rational values for r are encoded in the ratios between the numbers of teeth in the gear and the outer circle in which it rotates. Hypotrochoids are one small part of the immense topic of mathematical curves that can be drawn using mechanical apparatuses. For a sampling of past research in this area, see the Bridges papers by Craig [1], Sharp [2], and Tait [3].

A hypocycloid constructed from circles of radii 1 and p/q can be represented parametrically as follows:

$$x(\theta) = (1 - \frac{p}{q})\cos\theta + \frac{p}{q}\cos(\frac{q-p}{p}\theta) \quad (1)$$

$$y(\theta) = (1 - \frac{p}{q})\sin\theta - \frac{p}{q}\sin(\frac{q-p}{p}\theta) \quad (2)$$

I combine these two equations into a single parametric curve $H_{p,q}(\theta) = (x(\theta), y(\theta))$. From this definition it is clear that the hypocycloid is a closed curve with period $2\pi p$.

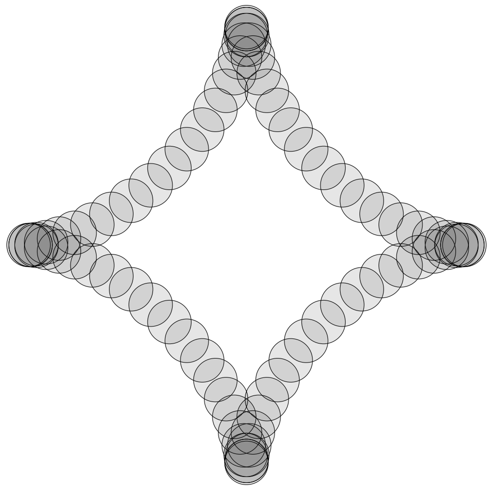

Figure 3 : *Visualization of a point moving around a rational hypocycloid with* $p = 1$ *and* $q = 4$ *(also known as an "astroid"). The drawing shows a superposition of 80 dots spaced evenly in parameter space from* $\theta = 0$ *to* $\theta = 2\pi$*. The bunching of the dots at the cusps shows that an animated point will slow down there.*

3 Moving points on hypocycloids

Let p and q be fixed. We can easily animate a point moving along a hypocycloid with "speed" s by drawing the point at position $H_{p,q}(st)$ at every time t. The point will move in even increments in parameter space, though certainly not at a constant speed in the plane: it will slow to a stop at each cusp and achieve maximum speed halfway between cusps (see Figure 3). Fortunately, this motion is naturalistic and visually elegant: a real-world object would need to slow to a stop in a similar manner in order to reverse direction at each cusp.

If we wish to animate multiple points travelling around the hypocycloid at the same time, the most natural approach is to space them evenly in the curve's parametric domain. To draw N evenly spaced points moving with constant speed s at a given time t, we can set $\theta_k = \frac{2\pi pk}{N} + st$ for $k = 0, \ldots, N-1$ and draw each point at position $P_k = H_{p,q}(\theta_k)$. Every hypocycloid juggling pattern is then defined from the 3-tuple (p, q, N) (animation speed is a not a fundamental aspect of the pattern).

Using the Processing programming environment (`https://processing.org/`), I created a visual interface that allows the user to adjust p, q and N interactively (together with a few other rendering parameters like the animation speed and radii of the points), and observe the resulting patterns that emerge. The patterns can be highly structured and symmetric—for example, when $N = q$, we obtain a symmetric dance of points that rush towards the centre of the large circle, twist around each other, and rush outwards again. Other patterns are far more chaotic, especially for large N. The most interesting patterns lie somewhere in between, when the combination of p, q and N yields hidden structure like that of Figure 1.

4 Wheels within wheels

By experimenting with the interactive program, it becomes clear that the spinning triangles in Figure 1(b) are not an isolated coincidence—many of these juggling patterns can be decomposed into a small number of "wheels", each one a copy of the gear bearing a regularly spaced set of points. Assuming that the speed s is positive, the individual points and the wheels travel counterclockwise about the centre of the large circle, while each wheel revolves clockwise about its centre. In this section I will explain how these configurations arise.

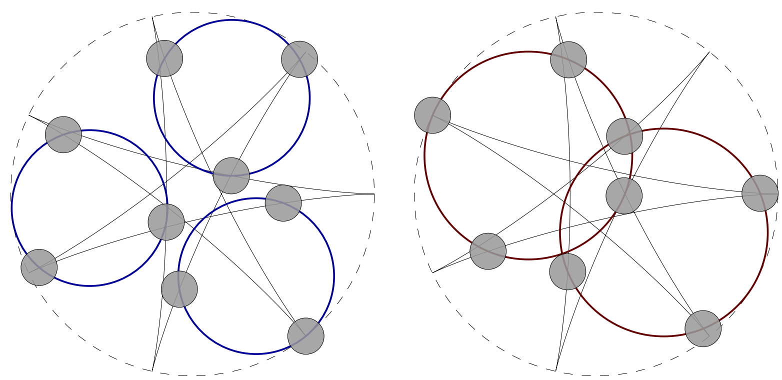

Figure 4 : *Two different juggling patterns with different groupings. On the left, the pattern* $(3,7,9)$ *decomposes into three primary wheels of three points each and no secondary wheels. On the right, the pattern* $(3,7,8)$ *decomposes into two secondary wheels of four points each and no primary wheels. In this figure and in Figure 6, the wheels are visualized using circles instead of regular polygons, in order to emphasize their connection to the underlying hypocycloid.*

Note that the equation for $H_{p,q}$ can be decomposed into two terms:

$$H_{p,q}(\theta) = (1-\frac{p}{q})(\cos\theta, \sin\theta) + \frac{p}{q}\left(\cos(\frac{q-p}{p}\theta), \sin(\frac{q-p}{p}\theta)\right)$$

In this formulation, the first term determines the position of the centre of the gear inside the large circle, and the second term determines the orientation of the gear. Thus I will say that the point $H_{p,q}(\theta)$ "lies on the wheel at θ" to mean that the gear's centre lies at $(1-\frac{p}{q})(\cos\theta, \sin\theta)$ at that point in the curve. In that light, the triangles in Figure 1(b) arise because multiple points lie on the same wheel.

Consider distinct integers j and k in the range $0,\ldots,N-1$. Using the definitions of the previous section, P_j and P_k will lie on the same wheel when $\theta_j = \frac{2\pi pj}{N} + st$ and $\theta_k = \frac{2\pi pk}{N} + st$ are congruent modulo 2π. This occurs when $\frac{p}{N}(j-k)$ is an integer, or when $pj \equiv pk (\mathrm{mod}\ N)$. Now, if p and N are relatively prime, no such congruences can occur and every point will lie on a distinct wheel. But in general, this equation can be rewritten as $j \equiv k(\mathrm{mod}\ N/\gcd(p,N))$, where gcd is the greatest common divisor function. It follows that the points will join together into $N/\gcd(p,N)$ distinct wheels, each one carrying $\gcd(p,N)$ points. With a bit more work, we can also see that the points will be equally spaced around each wheel. Returning to the example of Figure 1, which can at this point be revealed as a $(3,7,12)$ juggling pattern, we arrive at the mathematical fact that there will be $12/\gcd(3,12) = 4$ wheels, each containing $\gcd(3,12) = 3$ points.

5 A second family of wheels

The discussion in the previous section explains the blue triangles in Figure 1(b), but not the additional structure of the red squares in Figure 1(c). To be clear, the existence of one set of wheels does not imply the existence of the other—for example, the pattern $(3,7,9)$ can be seen as three rotating triangles of points but contains no equivalent to the red squares. Conversely, it turns out that the pattern $(3,7,8)$ decomposes into two wheels of four points each, in a way that is not predicted by the equations above. See Figure 4 for a comparison. These auxiliary red shapes move oppositely to the main ones: they travel clockwise around the centre of the large circle, while rotating counterclockwise about their own centres. How shall we predict the existence and structure of these wheels?

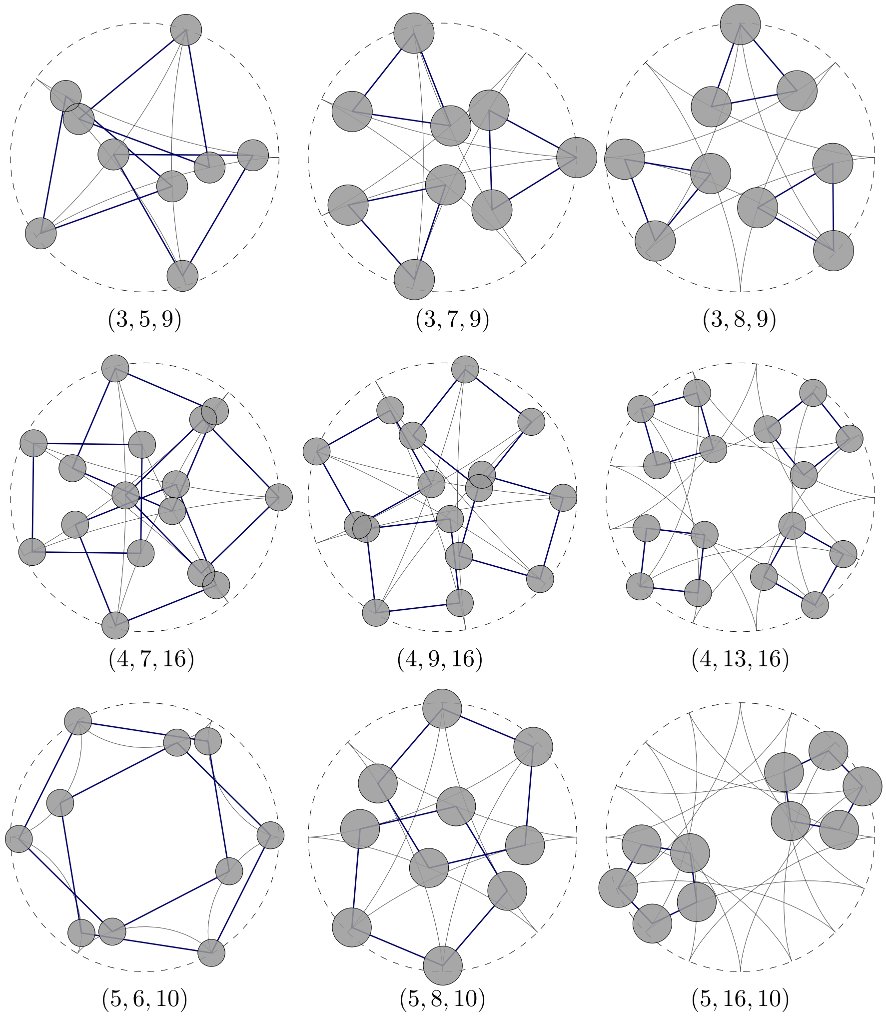

Figure 5 : *Examples of juggling patterns featuring only primary wheels and no secondary wheels. The examples are arranged to show rotating triangles in the first row, squares in the second row, and pentagons in the third row. As in Figure 1, the wheels are replaced by regular polygons to illustrate these arrangements.*

The auxiliary structure arises from an elementary fact about hypocycloids, one that I have not seen articulated elsewhere: every hypocycloid can be generated by *two* distinct small circles. If $H_{p,q}(\theta)$ is a given hypocycloid curve, then $H_{q-p,q}(\theta)$ will produce the same curve, albeit reparameterized. This relationship can be demonstrated rigorously via a simple change of parameter, deriving the identity $H_{p,q}(\theta) = H_{q-p,q}(\frac{p-q}{p}\theta)$.

In that case, the analysis given above applies equally well to this reparameterized hypocycloid. We arrive at an analogous conclusion: when the points are taken to lie on $H_{q-p,q}$, they can be decomposed into $N/\gcd(q-p,N)$ wheels, each one containing $\gcd(q-p,N)$ points. This relationship fills in the final piece of the puzzle that opened the paper: in the case of Figure 1 there must be $12/\gcd(7-3,12) = 3$ wheels, each containing $\gcd(7-3,12) = 4$ points. Having established that these two distinct families of wheels exist, I will refer to the wheels derived in the previous section as "primary wheels" and those shown to exist here as "secondary wheels".

We now have the tools we need to design interesting hypocycloid juggling patterns in which the points decompose into wheels. We start by choosing an integer q, and then choose $p > 1$ relatively prime to q. We can then choose any N that is the product of at least one factor of p and one factor of q (for example, $N = \mathrm{lcm}(p,q)$, the least

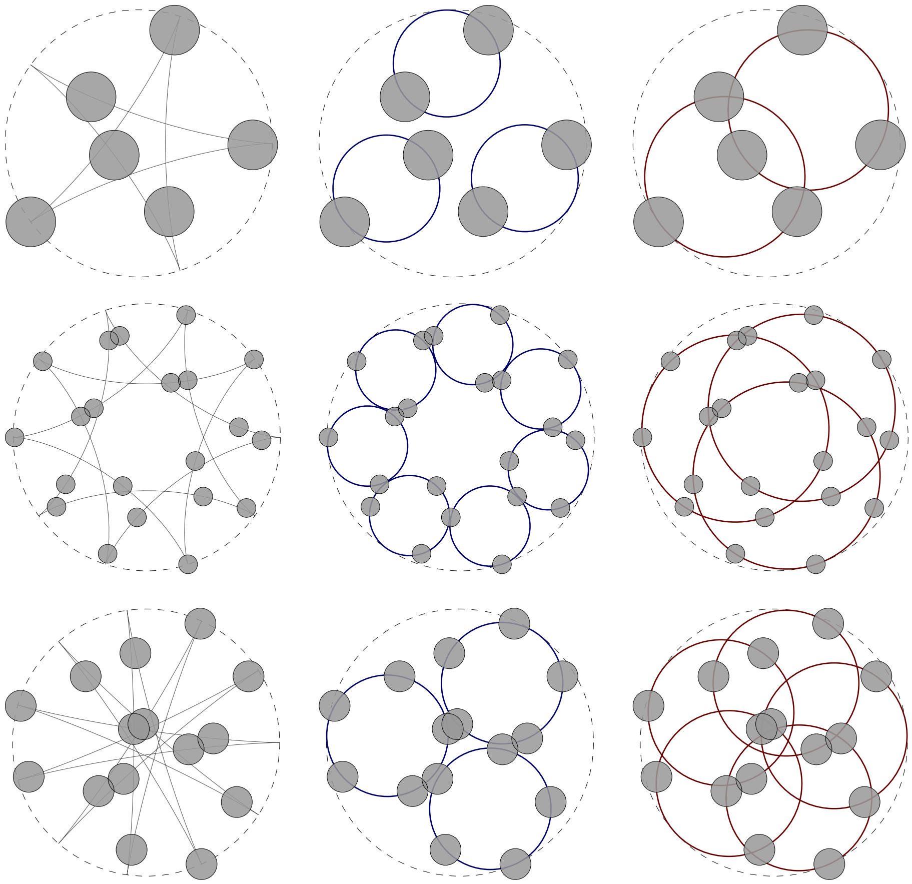

Figure 6: *Three examples of juggling patterns with non-trivial primary and secondary wheels. From top to bottom, the patterns are* $(2,5,6)$, $(3,10,21)$ *and* $(5,11,15)$. *Each row shows the dots and hypocycloid on the left, the primary wheels in the centre, and the secondary wheels on the right.*

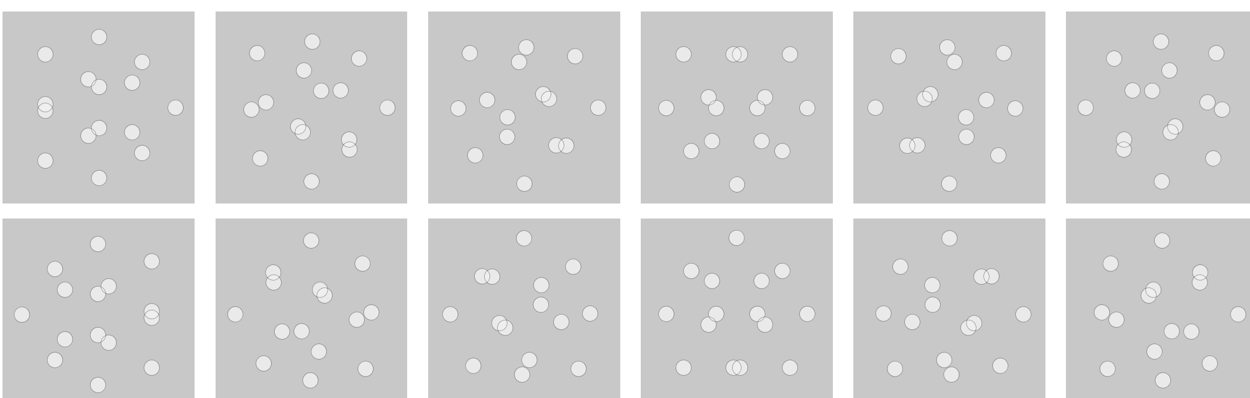

Figure 7: *12 frames that form an animated loop of the juggling pattern* $(3,8,15)$. *Notice that because of the symmetries of the underlying pattern, there are only two distinct configurations of points in these 12 frames; a bilaterally symmetric configuration appears in four rotations, and an asymmetric configuration appears in eight rotated and reflected orientations.*

common multiple of p and q). The pattern (p,q,N) will then necessarily contain non-trivial wheels rotating in both directions. Note that in such cases, the primary and secondary wheels will necessarily contain different numbers of points. For if $\gcd(p,N) = \gcd(q-p,N) > 1$, then p and $q-p$ must share a common factor, and hence so must p and q, contradicting their relative primality. Approaching the design problem from another perspective, we can always construct patterns consisting of a wheels of b points each—simply choose a pattern of the form (b,q,ab), where q is any number relatively prime to ab. When a and b are themselves relatively prime, then the pattern $(b,a+b,ab)$ will yield a primary wheels of b points each, and b secondary wheels of a points each.

6 Results and discussion

With the interactive software mentioned previously, it is easy to explore the large space of (p,q,N) patterns in search of especially appealing examples. I find that there are many worthwhile patterns to choose from, perhaps more so when N remains small. The rich family of patterns involving rotating wheels in one or two directions makes for a good starting point. Figure 5 shows some examples involving only primary wheels, and Figure 6 shows examples with both primary and secondary wheels. I also previously mentioned patterns of the form (p,q,q), leading to a q-fold symmetric pattern of rotating points. Another interesting family of patterns is those of the form $(1,q,q-2)$, in which the points seem to remain locked to an ellipse that rotates inside the large circle.

In addition to exporting static snapshots of the moving points, as seen in the figures in this paper, it is also possible to create looping animations. When doing so, we need to export only enough frames so that a given point P_k moves from its starting position to the starting position of P_{k+1}. The points are separated by the amount $\frac{2\pi p}{N}$ in parameter space. Thus, to create an animation loop with F frames, we draw the diagram at times $\frac{2\pi p}{N}\frac{i}{F}$ for $i = 0,\ldots,F-1$. Figure 7 shows an example of frames that form an animated loop for the pattern $(3,8,15)$. As always, the frames are far more compelling when stitched together into an actual animation.

The desire to create looping animations was the original inspiration for my investigation of this topic. In 2015 I taught a new introductory-level programming course for art students, using the Processing environment. At the time, Processing included a library for exporting animated GIFs, a popular contemporary medium for the expression of abstract mathematical animations (see for example the work of David Whyte at `http://beesandbombs.tumblr.com/`). The most successful animated GIFs are those with continuous periodic motion, in which the animation can run in an infinite loop with no obvious temporal discontinuities. In the search for a simple demonstration of periodic motion, I began with the motion of a single point around a regular polygon. I then extended my demonstration to star polygons and multiple points. To improve the visual quality of the results, I added "easing" to the points, having them decelerate into every polygon vertex and accelerate out again. At this point, the shape of the path and the non-uniform motion were sufficiently close to motion on hypocycloids that I began to notice the groupings into wheels. The wheels were not perfecty regular, but they were too close to regular to be a coincidence. I developed the mathematical ideas above both to explain what I was seeing, and to uncover the theoretically ideal simulation.

These animations represent a curious visual phenomenon. They should not be considered optical illusions—after all, the points really do assemble into primary and secondary wheels. Yet we can experience a similar sense of revelation when we suddenly perceive the fixed geometric relationships between points moving together. Our ability to assign these points to groups is known as the principle of "common fate" in Gestalt psychology. However, the groupings implied by the primary and secondary wheels seem to be visually incompatible: I find that I can attend to one set or the other, but not both simultaneously (and frequently, the two sets are not equally easy to pull out of the pattern). By focusing on the centre of the pattern, I also find that I can choose to ignore the groupings entirely and see the dancing points as a totality. It may be possible to tune the visibility of different structures in a pattern by experimenting with the rendering style, most obviously the radii of the moving points.

I can imagine a few geometric changes that would enhance the motion in these juggling patterns. It would be natural to experiment with superpositions of two or more juggling patterns, most obviously with the same values of q, but also potentially with different values. One might also scale and/or translate these multiple patterns to create larger animated figures. At that point, it would also be natural to experiment with nested wheels, in which each point is replaced by a rotating ring of smaller points. There may also be different ways to render the points in these patterns (e.g., colour coding) to bring out different effects, though I believe that the primary visual appeal lies in the unadorned abstract motion.

Ultimately, I believe these patterns would realize their full visual potential if constructed as mechanical devices, in which the points are affixed to the "planet gears" of a planetary gear train. Those planet gears would correspond to—and serve as visualizations of—the primary wheels in a (p,q,N) juggling pattern. However, the primary wheels often overlap, meaning that a straightforward planetary train might not suffice. Furthermore, if the primary wheels do not overlap, then the secondary wheels *must*, meaning that no simple apparatus can easily demonstrate both groupings of wheels simultaneously. It would therefore be especially interesting to develop a mechanical device that can show the moving points, the primary wheels, and the secondary wheels, all at the same time.

7 Conclusion

Beginning with a configuration of moving points with surprising structure, I have developed a mathematical framework based on hypocycloids, one that accounts for that structure and predicts a large class of related "juggling patterns". Many of these patterns offer a compelling visual experience, in which the points weave and dance around each other.

To be sure, many similar patterns could have been constructed directly, simply by drawing rotating wheels containing regularly spaced points. That being the case, hypocycloids might serve primarily as a means of organizing these patterns, or of discovering them in the first place. In my process of discovery, they played an essential role in clarifying a visual mystery: they explained the reason for the near-regular clusters of points I observed when moving points around star polygons. Or perhaps the mathematical novelty here is the insight we gain into hypocycloids based on the existence of these patterns.

Acknowledgments

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References

[1] Robert Craig. The mechanical drawing of cycloids, the geometric chuck. In Reza Sarhangi and John Sharp, editors, *Bridges London: Mathematics, Music, Art, Architecture, Culture*, pages 203–210, London, 2006. Tarquin Publications. Available online at `http://archive.bridgesmathart.org/2006/bridges2006-203.html`.

[2] John Sharp. Linkages to op-art. In Reza Sarhangi and John Sharp, editors, *Bridges London: Mathematics, Music, Art, Architecture, Culture*, pages 497–502, London, 2006. Tarquin Publications. Available online at `http://archive.bridgesmathart.org/2006/bridges2006-497.html`.

[3] Jack Tait. Taitographs: Drawings made by machines. In Reza Sarhangi and John Sharp, editors, *Bridges London: Mathematics, Music, Art, Architecture, Culture*, pages 403–410, London, 2006. Tarquin Publications. Available online at `http://archive.bridgesmathart.org/2006/bridges2006-403.html`.

[4] Wikipedia. Hypocycloid—Wikipedia, the free encyclopedia, 2016. Retrieved from `https://en.wikipedia.org/wiki/Hypocycloid` [Online; accessed 24-January-2016].