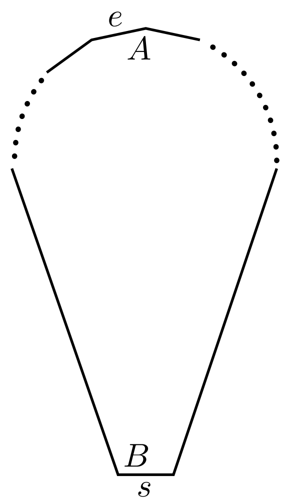
A quick sketch that the Ice Cream Cone has Heesch number 1

We consider a generic sketch of an ice cream cone with $n \geq 7$ sides, as shown below.



The "ball" is made from $n - 3$ short edges. The "cone" is made from two long edges, separated by one final short edge. The two corners at the base of the cone have interior angle B; all other interior angles are A. The goal is to glue copies of the base of the cone (marked s) onto every edge of the ball (a generic one is marked e), so that ball vertices are surrounded by one A and two Bs. Thus we have two equations relating A and B:

$$A + 2B = 360$$

$$(n-2)A + 2B = (n-2)180$$

Whence we can deduce

$$A = 180\frac{n-4}{n-3},\ B = 90\frac{n-2}{n-3}$$

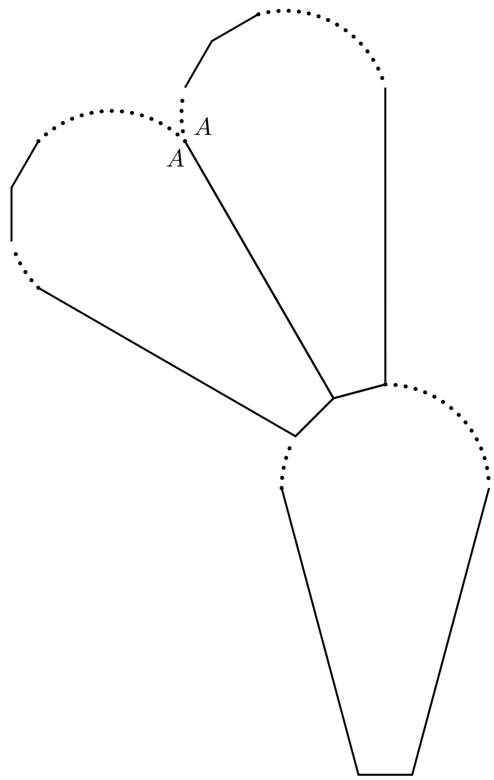
Note that for $n \geq 7$, we have $90 < A, B < 180$.

Now let us consider a generic ball edge, such as the one labelled e in the diagram. This edge is generic in the sense that it has another ball edge on each side. Note that because $n \geq 7$, there are at least two consecutive edges of this type in the shape.

Observe that any tile adjacent to e must abut it in a complete edge of the same length. In any other case, one of e's vertices will form a T-junction with an edge of the adjacent polygon. We would then be left with $360 - 180 - A$ degrees to fill in to surround the vertex. But $360 - 180 - A = 180/(n-3) \leq 90$, for all $n \geq 5$. Because A and B are both greater then $90°$, no part of the shape can be used to fill the remaining space.

Thus, e must abut either another whole ball edge, or a copy of the base edge s. But in the former case, at least one of e's vertices will then have two A vertices next to it, leaving $360 - 2A = 360/(n-3)$ of empty space to fill in. But $360/(n-3) \leq 90$ for $n \geq 7$, which would again leave the vertex unsurroundable.

We conclude that edges like e must be adjacent to cone base edges. But there will be at least two consecutive e edges in the shape, leading to the following partial surround:



Observe that the two copies glued onto e edges must necessarily produce a new vertex partially surrounded by two A vertices, which we have already seen can not be fully surrounded. Thus the ice cream cone can be surrounded once but not twice, and has Heesch number 1.